Announcements

1) Quiz Thursday

Back to the Heat Equation

Two - Dimensional:

$u = u(x, y, t)$ the temperature in a 2-D region

$$\boxed{k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}}$$

is the heat equation, where again, k is the thermal conductivity of the material.

Time Independence

We assume that the value $u(x,y,t)$ does not depend on t $\Rightarrow$ $\frac{\partial u}{\partial t} = 0$.

The time independent heat equation is

$$\boxed{k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0}$$

Circular Domains

We measure the heat flow on a disk of radius R, so the region is

$$\boxed{D = \{(r,\theta) \mid 0 \le r \le R,\ 0 \le \theta < 2\pi\}}$$

Boundary conditions on u of the form

$$u(R,\theta) = f(\theta)$$

(boundary circle of radius R)

Switch to Polar Coordinates

Recall we have

$$r = \sqrt{x^2 + y^2} \; ,$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

Can then write

$$u = u(r, \theta)$$ (t suppressed since u is time-independent).

Rewrite the heat equation in polar coordinates!

We want $\dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 u}{\partial y^2}$ in terms of r and θ. Use the Chain Rule!

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\frac{\partial}{\partial x}\left(\left(x^2+y^2\right)^{1/2}\right) +$$

$$\frac{\partial u}{\partial \theta}\frac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right)\right)$$

$$= \frac{\partial u}{\partial r}\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\left(\frac{-y}{x^2}\right)\frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$= \boxed{\frac{\partial u}{\partial r}\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right)}$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r}\frac{\partial}{\partial y}\left((x^2+y^2)^{1/2}\right) +$$

$$\frac{\partial u}{\partial \theta}\frac{\partial}{\partial y}\left(\arctan\left(\frac{y}{x}\right)\right)$$

$$= \frac{\partial u}{\partial r}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\left(\frac{1}{x}\right)\left(\frac{1}{1+\left(\frac{y}{x}\right)^2}\right)$$

$$= \boxed{\frac{\partial u}{\partial r}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\frac{x}{x^2+y^2}}$$

Extra Credit (10 points, due Tuesday after exam 2)

Show that, in polar coordinates, the heat equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{becomes}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

by using the chain rule to compute $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Example 1: Attempt to solve

the heat equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

on the unit disk

$$D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta < 2\pi\}$$

with $u(1,\theta) = 1$.

Suppose

$$u(r,\theta) = f(r)\,g(\theta)$$

for some real-valued functions

f and g.

$$\frac{\partial u}{\partial r} = f'(r)\,g(\theta),$$

$$\frac{\partial^2 u}{\partial r^2} = f''(r)\,g(\theta),$$

$$\frac{\partial^2 u}{\partial \theta^2} = f(r)\,g''(\theta)$$

Plug into heat equation:

$$f''(r)g(\theta) + \frac{1}{r}f'(r)g(\theta) + \frac{1}{r^2}f(r)g''(\theta)$$

$$= 0$$

Multiply through by r^2:

$$r^2 f''(r)g(\theta) + rf'(r)g(\theta) + f(r)g''(\theta) = 0,$$

so

$$\boxed{r^2 f''(r)g(\theta) + rf'(r)g(\theta) = -f(r)g''(\theta)}$$

Dividing by $f(r)g(\theta)$, we get

$$\boxed{\frac{r^2 f''(r) + r f'(r)}{f(r)} = \frac{-g''(\theta)}{g(\theta)}}$$

This means there must be a constant α such that

$$\boxed{\frac{r^2 f''(r) + r f'(r)}{f(r)} = \alpha = \frac{-g''(\theta)}{g(\theta)}}$$

We get two differential equations

$$\boxed{\begin{gathered} r^2 f''(r) + r f'(r) + \alpha f(r) = 0 \\ g''(\theta) + \alpha g(\theta) = 0 \end{gathered}}$$

2^{nd} equation: second order, constant coefficients, homogeneous,

so one solution is

$$g(\theta) = e^{a\theta}$$

1^{st} Equation: 2^{nd} order, homogeneous,

nonconstant coefficients — it is

a Cauchy-Euler equation!

One solution given by

$$f(r) = r^s$$

Solve:

$$r^2(s)(s-1)r^{s-2} + r(s)r^{s-1}$$

$$+ \alpha r^s = 0, \text{ so}$$

$$r^s(s^2 \cancel{-s+s} + \alpha) = 0$$

We have

$$s^2 + \alpha = 0,$$

$$s = \pm\sqrt{-\alpha} \quad \text{(more difficult if } \alpha > 0\text{)}$$

$\alpha < 0$, $r^{\sqrt{-\alpha}}$, $r^{-\sqrt{-\alpha}}$ are

two linearly independent solutions,

so

$$\boxed{f(r) = c_1 r^{\sqrt{-\alpha}} + c_2 r^{-\sqrt{-\alpha}}}$$

More on Electrical Circuits

(Section 5.7)

Mea Culpa on Capacitors

In a simple RC circuit,

$$E(t) = E_C + E_R$$

where $E_R = RI$, $E_C = \frac{q}{C}$.

q is nonconstant!

$$\frac{dq}{dt} = I(t), \quad \text{so}$$

we may rewrite

$$\boxed{R\frac{dq}{dt} + \frac{q}{C} = E(t)}$$